1. Prove that 7 is the only prime number that precedes a perfect cube. A perfect cube is a number x ∈ N such that there exists n ∈ N and x = n3. Rewrite the statement using an implication and prove the statement’s correctness.

Let P(x) be x is prime.

For x to be prime, one of the factors must be equal to 1:

Let n - 1 = 1, therefore n = 2

If n = 2,

Therefore x = 7 ∎

1. A Test for Primality is the following:

Given an integer n > 1, to test whether n is prime check to see if it is divisible by a prime number less than or equal to its square root. If it is not divisible by any of these numbers then it is prime.

We will show that this is a valid test.

Contrapositive:

By the fundamental theorem of arithmetic, all positive non-prime integers can be expressed as a product of primes: this satisfies p is prime and p | n.

Let n = xy, where x and y are integer factors of n.

Let :

Suppose :

Then: and

Which means

This contradicts n = xy

Therefore ∎

1. State the contrapositive of the statement of part (b).
2. Prove that for all natural numbers n, n is either a perfect square or the square root of n is irrational.

Assume to the contrary that , where n is not a perfect square:

, where is in its most simplified form.

q also cannot be equal to one, as that would make an integer, which means it is a perfect square.

Because cannot be equal to 1, and is in its lowest form, n is not an integer.

This contradicts that n is an element of natural numbers.

Therefore ∎

1. The greatest common divisor c, of a and b, denoted as c = gcd(a, b), is the largest number that divides both a and b. One way to write c is as a linear combination of a and b. Then c is the smallest natural number such that c = ax+by for x, y ∈ Z. We say that a and b are relatively prime iff gcd(a, b) = 1. Prove that a and n are relatively prime if and only if there exists integer s such that sa ≡n 1. We call s the inverse of a modulo n.

Prove

Prove:

Contrapositive:

Let s = x and = y, because the difference between arbitrary integers is an integer.

Prove:

Let s = x and = y, because the difference between arbitrary integers is an integer.

∎

1. Use simple induction to prove that for all n ≥ 1,

Base case, n = 1:

(1+1)! – 1 = 1

Assume S(k) is true for some k ≥ 1

S(k):

Induction step, prove: S(k+1) = (k + 2)! – 1

∎

1. Suppose that n girls and n boys are distributed around the outside of a circular table. Use mathematical induction to prove that for any integer n ≥ 1, given any such seating plan, it is always possible to find a starting point so that if travel around the table in a clockwise direction the number of girls you pass is never less than the number of boys you have passed. For example, in the diagram below, we use g to denote a girl and b to denote a boy, you should start at the girl in red.

Base case, n = 1:

The only possible orders are gb or bg, in which gb satisfies that the number of g will always be greater than or equal to the number of b’s passed.

Hypothesis:

Let S(k) be that there is starting point which the number of g’s is always greater than or equal to the number of b’s for some k ≥ 1 for which there are k number of b’s and k number of g’s.

Induction:

Prove S(k+1)

This means there is one more g and b in the circle. Because you need to pass all b’s and g’s in the circle, there must be, at some point, a g preceding a b. If we take these out, we have the case of S(k) which is true because of the induction hypothesis. If you add back the g preceding a b, then S(k+1) is true, as no matter what S(k) is, adding a g preceding a b will not change the starting point as the increase in the number of b’s passed is the same as the number g’s passed.

∎